Linear programing

Tutorial 9

Dual Simplex method and Dual Problem

Dual Problem:

max	$\sum_{j=1}^{n} c_j x_j$	min	$\sum_{i=1}^{m} u_i b_i$
subject to	$\sum_{j=1}^{n} a_{ij} x_j \le b_i \ (i = 1, 2, \cdots, k)$	subject to	$u_i \ge 0 \ (i = 1, 2, \cdots, k)$
	$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = k+1, \cdots, m)$		u_i free $\quad (i = k+1, \cdots, m)$
	$x_j \ge 0 \quad (j = 1, 2, \cdots, \ell)$		$\sum_{i=1}^{m} u_j a_{ij} \ge c_j \quad (j = 1, 2, \cdots, \ell)$
	x_j free $\quad (j = \ell+1, \cdots, n)$		$\sum_{i=1}^{m} u_i a_{ij} = c_j \quad (j = \ell+1, \cdots, n)$

Key Concepts:

1) Number of variables in the llp = number of constrains in the dual
2) If the constrain matrix of the llp is A, the the constrain matrix of the dual is A^T

Theorem 5.2 (Weak Duality Theorem). *If* x *is a feasible solution (not necessarily basic) to the primal and* u *is a feasible solution (not necessarily basic) to the dual, then*

$$\mathbf{c}^T \mathbf{x} \le \mathbf{b}^T \mathbf{u} .$$

Theorem 5.3. *If* $\mathbf{x}_0$ *and* $\mathbf{u}_0$ *are feasible solutions to the primal and the dual respectively and if*

$$\mathbf{c}^T \mathbf{x}_0 = \mathbf{b}^T \mathbf{u}_0,$$

then $\mathbf{x}_0$ *and* $\mathbf{u}_0$ *are optimal solutions to the primal and the dual respectively.*

Theorem 5.4 (The Strong Duality Theorem). *A feasible solution* $\mathbf{x}_0$ *to the primal is optimal if and only if there exists a feasible solution* $\mathbf{u}_0$ *to the dual such that*

$$\mathbf{c}^T \mathbf{x}_0 = \mathbf{b}^T \mathbf{u}_0 . \tag{5.2}$$

In particular, $\mathbf{u}_0$ *is an optimal solution to the dual.*

Example 1: A case that is too good to be true

Consider the following LPP,

$$\begin{aligned} \text{maximize} \quad & z = x_1 + 5x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 = 3 \\ & 2x_1 - x_2 = 4 \\ & x_1,\ x_2,\ x_3 \geqslant 0 \end{aligned}$$

Given that the optimal basic variables are x_1 and x_3, determine the associated optimal dual solution.

The constraints in the dual problem are:

min $3u_1 + 4u_2 = Z_0$

$$u_1 + 2u_2 \geq 1$$

$$2u_1 - u_2 \geq 5$$

$$u_1 \geq 3$$

$$u_i is free$$

X_1 and X_3 are basic, i.e solution takes the from $(X_1, 0, X_3)$.

$X_2 = 0$ $\quad X_1 + X_3 = 3$, $2X_1 = 4$ $\Rightarrow X_1 = 2 \quad X_3 = 1 \Rightarrow Z = 5$

$U_1 = 3$, $U_2 = -1$ $\quad Z_0 = 5$

Theorem 5.7 (Complementary Slackness). *Given any pair of optimal solutions to an LP problem and its dual, then*

(i) for each i, $i = 1, 2, \cdots, m$, the product of the ith primal slack variable and ith dual variable is zero, and

(ii) for each j, $j = 1, 2, \cdots, n$, the product of the jth primal variable and jth surplus dual variable is zero.

Example 2:

Consider the following LPP.

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 5x_2 - x_3 + 2x_4 - 4x_5 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + 3x_4 + x_5 \leqslant 6 \\ & -x_1 - x_2 + 2x_3 + x_4 - x_5 \geqslant 3 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geqslant 0 \end{aligned}$$

(a) Write down the dual problem.

The dual problem is :
Max

$$6u_1 + 3u_2$$

subject to

$$\begin{aligned} u_1 - u_2 &\leq 3 \\ u_1 - u_2 &\leq 5 \\ u_1 + 2u_2 &\leq -1 \\ 3u_1 + u_2 &\leq 2 \\ u_1 - u_2 &\leq -4 \\ u_1 \leq 0,\ u_2 &\geq 0 \end{aligned}$$

(b) If the point (-3,1) is the optimal solution to the dual problem, find the optimal solution to the initial problem.

Since x_1 and x_3 are basic variables then in equalities in the first and it the second constraint hold. Solving them, we get $u_1 = 3$ and $u_2 = -1$.

3rd and 5th contrainty holds, i.e X_3, X_5 in the primal problem are basic, ie $X_3 = X_5 = 3$.

Example 3:

Consider the following LPP

$$\begin{aligned} \text{maximize} \quad & 6x_1 + 7x_2 + 3x_3 + 2x_4 + x_5 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 6 \\ & 2x_1 + 3x_2 + 4x_2 + x_5 = 14 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geqslant 0 \end{aligned}$$

(a) Find the corresponding dual problem
(b) If x_1 and x_5 are basic in the optimal solution of the lpp, then find the optimal solutino for the dual

The dual problem is :
Min

$$6u_1 + 14u_2$$

subject to

$$\begin{aligned} u_1 + 2u_2 &\geq 6 \\ u_1 + 3u_2 &\geq 7 \\ u_1 + 4u_2 &\geq 3 \\ u_1 &\geq 2 \\ u_2 &\geq 1 \\ u_1, u_2 is free \end{aligned}$$

(b)
Using constraint 1 and 5 in the dual problem,that is

$$\begin{aligned} u_1 + 2u_2 &= 6 \\ u_2 &= 1 \end{aligned}$$

we have $u_1 = 4, u_2 = 1$

Dual Simplex method:

Algorithm for the dual simplex method

1. Given a *dual* BFS $\mathbf{x}_B$, if $\mathbf{x}_B \geq 0$, then the current solution is optimal; otherwise select an index r such that the component x_r of $\mathbf{x}_B$ is negative.

2. If $y_{rj} \geq 0$ for all $j = 1, 2, \cdots, n$, then the dual is unbounded; otherwise determine an index s such that

$$-\frac{y_{0s}}{y_{rs}} = \min_j \left\{ -\frac{y_{0j}}{y_{rj}} \middle| y_{rj} < 0 \right\} .$$

3. Pivot at element y_{rs} and return to step 1.

Theorem 5.5. *If B is the basis matrix for the primal corresponding to an optimal solution and $\mathbf{c}_B$ contains the prices of the variables in the basis, then an optimal solution to the dual is given by $(B^{-1})^T\mathbf{c}_B$, i.e., the entries in the x_0 row under the columns corresponding to the slack variables give the values of the dual structural variables. Moreover, the entries in the x_0 row under the columns for the structural variables will give the optimal values of the dual surplus variables.*

Recall example from lecture:

$$\max \quad x_0 = 4x_1 + 3x_2$$

$$\text{subject to} \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 8 \\ 7 \\ 15 \\ 1 \end{bmatrix} \\ x_1, x_2 \geq 0. \end{cases}$$

5 slack variables correspond to the 5 variable (5U) in the dual.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
x_3	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_2	0	1	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	3
x_4	0	0	0	1	$\frac{3}{2}$	$\frac{1}{2}$	0	5
x_1	1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	4
x_7	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	4
x_0	0	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	0	25

Thus the optimal solution is $[x_1, x_2] = [4, 3]$ with $[x_3, x_4, x_5, x_6, x_7] = [2, 5, 0, 0, 4]$. From the x_0 row, we see that the optimal solution to the dual is given by

$$[u_1, u_2, u_3, u_4, u_5, u_6, u_7] = \left[0, 0, \frac{5}{2}, \frac{1}{2}, 0, 0, 0\right].$$

Example 4:

Solve the following LPPs by dual simplex method and find out the optimal values of all the primal and dual variables

$$\min \quad 2x_1 + x_2 + x_3$$

$$x_1 + x_2 \leq 3$$
$$x_1 - 2x_2 \geq 1$$
$$x_2 + x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

(a)

(1)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	1	1	0	1	0	0	3
x_5	-1	2	0	0	1	0	-1
x_6	0	-1	-1^*	0	0	1	-4
x_0	2	1	1	0	0	0	0

(2)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	1	1	0	1	0	0	3
x_5	-1^*	2	0	0	1	0	-1
x_3	0	1	1	0	0	-1	4
x_0	2	0	0	0	0	1	-4

(3)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	0	3	0	1	1	0	2
x_1	1	-2	0	0	-1	0	1
x_3	0	1	1	0	0	-1	4
x_0	0	4	0	0	2	0	-6

Thus optimal solution is (1,0,4) with the dual solution (0,2,0)

Example 5:

$$\min \quad 2x_1 + x_2$$

$$x_1 + x_2 \leq 5$$
$$7x_1 - x_2 \geq 21$$
$$x_1 \geq 4$$
$$x_1, x_2 \geq 0$$

(1)	x_1	x_2	x_3	x_4	x_5	b
x_3	1	1	1	0	0	5
x_4	-7	1	0	1	0	-21
x_5	1	0	0	0	1	-4
x_0	2	1	0	0	0	0

Since the row of x_5 has no negative element except in the column of b, thus the dual is unbounded which means that the primal problem has no feasible solution.